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Best Simultaneous Approximation in $L^p(I, E)$

Fathi B. Saidi and Deeb Hussein[1]

Department of Mathematics, University of Sharjah, P. O. Box 27272, Sharjah, United Arab Emirates
E-mail: fsaidi@sharjah.ac.ae

and

Roshdi Khalil

Department of Mathematics, University of Jordan, Amman, Jordan
E-mail: roshdi@ju.edu.jo

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Let G be a reflexive subspace of the Banach space E and let $L^p(I, E)$ denote the space of all p-Bochner integrable functions on the interval $I = [0, 1]$ with values in E, $1 \leqslant p \prec \infty$. Given any norm $N(\cdot, \cdot)$ on R^2, N nondecreasing in each coordinate on the set R^2_+, we prove that $L^p(I, G)$ is N-simultaneously proximinal in $L^p(I, E)$. Other results are also obtained.

Key Words: simultaneous; approximation.

1. INTRODUCTION

Throughout this paper, E is a Banach space, G is a closed subspace of E, p is a real number in $[1, \infty)$, and n is any integer, $n \geqslant 1$. The norm of $v \in E$ is denoted by $\|v\|$ and the norm of $u := (u_k)_{k=1}^n \in E^n$ is defined by

$$\|u\|_{p,n} := \left[\sum_{k=1}^{n} \|u_k\|^p \right]^{1/p}.$$

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Also, we let $L^p(I, E)$ be the Banach space of p-Bochner integrable functions defined on I with values in E, where $I = [0, 1]$ is the unit interval in R. Here R is the set of real numbers. The norm of $f \in L^p(I, E)$ is given by

$$\|f\|_p := \left[\int_I \|f(s)\|^p \, d\mu \right]^{1/p},$$

where μ is the Lebesgue measure on I.

Finally, we let N be any norm on R^2 satisfying, for every (x_1, x_2), $(y_1, y_2) \in R^2$,

$$N(x_1, x_2) \leqslant N(y_1, y_2), \qquad \text{if} \quad |x_i| \leqslant |y_i|, \qquad i = 1, 2. \tag{1.1}$$

Note that Eq. (1.1) is equivalent to N is nondecreasing in each coordinate on the set $R^2_+ := \{(x_1, x_2) : x_1, x_2 \geqslant 0\}$. Also, note that Eq. (1.1) is satisfied by all the l^p-norms on R^2, $1 \leqslant p \leqslant \infty$.

The norm of $(u^1, u^2) \in (E^n)^2$ is defined by

$$\begin{aligned} |(u^1, u^2)|_{p,n} &:= N\left(\left[\sum_{k=1}^{n} \|u_k^1\|^p \right]^{1/p}, \left[\sum_{k=1}^{n} \|u_k^2\|^p \right]^{1/p} \right) \\ &:= N(\|u^1\|_{p,n}, \|u^2\|_{p,n}), \end{aligned}$$

where $u^1 = (u_k^1)_{k=1}^n$, $u^2 = (u_k^2)_{k=1}^n$. Note that, by Eq. (1.1), $|\cdot|_{p,n}$ is a norm on $(E^n)^2$ making it a Banach space. The diagonal of G^n is given by

$$D^n := \{((g_k)_{k=1}^n, (g_k)_{k=1}^n) : (g_k)_{k=1}^n \in G^n\}.$$

DEFINITION 1. We say that $g \in G$ is a best N-simultaneous approximation from G of the pair of elements $u^1, u^2 \in E$ if, for every $h \in G$,

$$N(\|u^1 - g\|, \|u^2 - g\|) \leqslant N(\|u^1 - h\|, \|u^2 - h\|),$$

or, in other words, if, for every $h \in G$,

$$|(u^1 - g, u^2 - g)|_{1,1} \leqslant |(u^1 - h, u^2 - h)|_{1,1}.$$

Note that $g \in G$ is a best N-simultaneous approximation from G of u^1, $u^2 \in E$ if and only if (g, g) is a best approximation from D^1 of the pair $(u^1, u^2) \in E^2$, where the norm on E^2 is $|\cdot|_{1,1}$. If every pair of elements u^1, $u^2 \in E$ admits a best N-simultaneous approximation from G (equivalently, D^1 is proximinal in E^2), then G is said to be N-simultaneously proximinal in E.

The problem of best simultaneous approximations has been studied by many authors, e.g., [1, 7, 13–15]. Most of these works have dealt with the characterizations of best simultaneous approximations in spaces of continuous functions with values in a Banach space E. Some existence and uniqueness results were also obtained. Results on best simultaneous approximation in general Banach spaces may be found in [9] and [11]. Little or no work has been done in the spaces $L^p(I, E)$. It is the aim of this paper to establish some existence results in this area. Among other things, we prove that, if G is a reflexive subspace of the Banach space E and $1 \leqslant p < \infty$, then $L^p(I, G)$ is N-simultaneously proximinal in $L^p(I, E)$.

Before we continue we note, as pointed out in Definition 1, that problems of best simultaneous approximation can also be viewed as special cases of vector-valued approximation. Some recent work in this area is due to Pinkus [10].

2. BEST SIMULTANEOUS APPROXIMATION IN $L^P(I, E)$

Recall that the norm of $u \in E^n$, hence also of $u \in G^n$ and of $u \in D^n$, is $\|u\|_{p,n}$ where p is a fixed real number in $[1, \infty)$.

We start this section with the following observations:

Remark 1. Since all norms on a finite dimensional vector space are equivalent and since $N([\sum_{k=1}^n (\cdot)^p]^{1/p}, [\sum_{k=1}^n (\cdot)^p]^{1/p})$ is a norm on $R^n \times R^n$, we have (where the norm on $(E^n)^2$ is $|\cdot|_{p,n}$):

(1) A sequence $((u_k^m)_{k=1}^n, (v_k^m)_{k=1}^n)_{m=1}^\infty$ in $(E^n)^2$ is bounded if and only if the sets $\{u_k^m : 1 \leqslant k \leqslant n, 1 \leqslant m < \infty\}$ and $\{v_k^m : 1 \leqslant k \leqslant n, 1 \leqslant m < \infty\}$ are both bounded in E.

(2) A sequence $((u_k^m)_{k=1}^n, (v_k^m)_{k=1}^n)_{m=1}^\infty$ in $(E^n)^2$ is convergent (weakly convergent) if and only if all the sequences $(u_k^m)_{m=1}^\infty$, $(u_k^m)_{m=1}^\infty$, $1 \leqslant k \leqslant n$, are convergent (weakly convergent) in E.

(3) If G is reflexive then G^n and D^n are reflexive.

Note that $(g_k)_{k=1}^n \in G^n$ is a best N-simultaneous approximation from G^n of the pair of elements $(u_k^1)_{k=1}^n$, $(u_k^2)_{k=1}^n \in E^n$ if and only if, for every $(h_k)_{k=1}^n \in G^n$,

$$|((u_k^1 - g_k)_{k=1}^n, (u_k^2 - g_k)_{k=1}^n)|_{p,n} \leqslant |((u_k^1 - h_k)_{k=1}^n, (u_k^2 - h_k)_{k=1}^n)|_{p,n}.$$

It follows immediately that

Remark 2. Let $(g_k)_{k=1}^n \in G^n$ be a best N-simultaneous approximation from G^n of the pair of elements $(u_k^1)_{k=1}^n$, $(u_k^2)_{k=1}^n \in E^n$. Then, for each $k \in \{1, ..., n\}$, $g_k = 0$ whenever $u_k^1 = u_k^2 = 0$.

From Remark 1 we obtain that, if G is reflexive, then D^n is reflexive and, consequently, proximinal in $(E^n)^2$. Hence, we have:

LEMMA 1. *If G is reflexive then, for every $n \geqslant 1$, G^n is N-simultaneously proximinal in E^n.*

Now, note that $g \in L^p(I, G)$ is a best N-simultaneous approximation from $L^p(I, G)$ of $f_1, f_2 \in L^p(I, E)$ if and only if, for all $h \in L^p(I, G)$,

$$N(\|f_1 - g\|_p, \|f_2 - g\|_p) \leqslant N(\|f_1 - h\|_p, \|f_2 - h\|_p).$$

The main result of this section is:

THEOREM 1. *If, for every $n \geqslant 1$, G^n is N-simultaneously proximinal in E^n, then every pair of simple functions $f_1, f_2 \in L^p(I, E)$ admits a best N-simultaneous approximation g from $L^p(I, G)$.*

Proof. Let f_1, f_2 be two simple functions in $L^p(I, E)$. Then $f_j(s) := \sum_{k=1}^n u_k^j \chi_{I_k}(s)$, $j = 1, 2$, where the I_k's are disjoint measurable subsets of I satisfying $\bigcup_{k=1}^n I_k = I$ and χ_{I_k} is the characteristic function of I_k. Since f_1 and f_2 represent classes of functions, we may assume that $\mu(I_k) > 0$, $1 \leqslant k \leqslant n$. By assumption, there exists an N-simultaneous best approximation $(w_k)_{k=1}^n$ from G^n of the pair of elements $(\mu^{1/p}(I_k)\, u_k^1)_{k=1}^n, (\mu^{1/p}(I_k)\, u_k^2)_{k=1}^n \in E^n$. This implies, if $g_k := \frac{1}{\mu^{1/p}(I_k)} w_k$, that $g := \sum_{k=1}^n g_k \chi_{I_k} \in L^p(I, G)$ and, since $w_k = \mu^{1/p}(I_k)\, g_k$, that

$$\begin{aligned} &|((\mu^{1/p}(I_k)(u_k^1 - g_k))_{k=1}^n, (\mu^{1/p}(I_k)(u_k^2 - g_k))_{k=1}^n)|_{p,n} \\ &\quad \leqslant |((\mu^{1/p}(I_k)(u_k^1 - h_k))_{k=1}^n, (\mu^{1/p}(I_k)(u_k^2 - h_k))_{k=1}^n)|_{p,n} \end{aligned} \tag{2.1}$$

for all $h := \sum_{k=1}^n h_k \chi_{I_k} \in L^p(I, G)$. In other words, we have

$$N(\|f_1 - g\|_p, \|f_2 - g\|_p) \leqslant N(\|f_1 - h\|_p, \|f_2 - h\|_p), \tag{2.2}$$

for all $h := \sum_{k=1}^n h_k \chi_{I_k} \in L^p(I, G)$. We need to show that Eq. (2.2) holds for all simple functions (hence, by density, for all functions) $h \in L^p(I, G)$. So let h be any simple function in $L^p(I, G)$. Then $h := \sum_{i=1}^m h_i \chi_{J_i}(s)$, where the J_i's are disjoint, $\bigcup_{i=1}^m J_i = I$. Then

$$f_j = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant k \leqslant n}} u_{ki}^j \chi_{I_k \cap J_i}, \qquad j = 1, 2, \qquad \text{and} \qquad h = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant k \leqslant n}} h_{ki} \chi_{I_k \cap J_i},$$

where, for each j and each k, $j = 1, 2$ and $1 \leqslant k \leqslant n$,

$$u_{ki}^j = u_k^j, \qquad 1 \leqslant i \leqslant m, \tag{2.3}$$

and, for each i, $1 \leqslant i \leqslant m$,

$$h_{ki} = h_i, \qquad 1 \leqslant k \leqslant n.$$

Again, we obtain from the assumption that there exists a best N-simultaneous approximation $(w^*_{ki})^{1 \leqslant i \leqslant m}_{1 \leqslant k \leqslant n}$ from G^{nm} of the pair of elements $(\mu^{1/p}(I_k \cap J_i) u^1_{ki})^{1 \leqslant i \leqslant m}_{1 \leqslant k \leqslant n}$, $(\mu^{1/p}(I_k \cap J_i) u^2_{ki})^{1 \leqslant i \leqslant m}_{1 \leqslant k \leqslant n} \in E^{nm}$. Note that, by Remark 2, $w^*_{ki} = 0$ whenever $\mu^{1/p}(I_k \cap J_i) u^1_{ki} = \mu^{1/p}(I_k \cap J_i) u^2_{ki} = 0$. Let $w^*_{ki} := \mu^{1/p}(I_k \cap J_i) g^*_{ki}$, $g^*_{ki} := 0$ if $w^*_{ki} = 0$. Then,

$$g^* := \sum_{1 \leqslant k \leqslant n}^{1 \leqslant i \leqslant m} g^*_{ki} \chi_{I_k \cap J_i} \in L^p(I, G)$$

and

(2.4)

$$\begin{aligned} &|((\mu^{1/p}(I_k \cap J_i)(u^1_{ki} - g^*_{ki}))^{n,m}_{k,i=1}, (\mu^{1/p}(I_k \cap J_i)(u^2_{ki} - g^*_{ki}))^{n,m}_{k,i=1})|_{p,nm} \\ &\qquad \leqslant |((\mu^{1/p}(I_k \cap J_i)(u^1_{ki} - h_{ki}))^{n,m}_{k,i=1}, (\mu^{1/p}(I_k \cap J_i)(u^2_{ki} - h_{ki}))^{n,m}_{k,i=1})|_{p,nm}. \end{aligned}$$

Note that, if $\lambda_{ki} := \mu(I_k \cap J_i)/\mu(I_k)$, then $\sum_{i=1}^m \lambda_{ki} = 1$ and

$$\sum_{1 \leqslant k \leqslant n}^{1 \leqslant i \leqslant m} \mu(I_k \cap J_i) \|u^j_{ki} - g^*_{ki}\|^p = \sum_{k=1}^n \mu(I_k) \sum_{i=1}^m \lambda_{ki} \|u^j_{ki} - g^*_{ki}\|^p, \qquad j = 1, 2.$$

Therefore, since $\|v\|^p$ is a convex function of $v \in E$ for $p \geqslant 1$,

$$\sum_{i=1}^m \lambda_{ki} \|u^j_{ki} - g^*_{ki}\|^p \geqslant \left\| \sum_{i=1}^m \lambda_{ki} u^j_{ki} - \sum_{i=1}^m \lambda_{ki} g^*_{ki} \right\|^p = \|u^j_k - \beta_k\|^p, \qquad j = 1, 2,$$

where $\beta_k := \sum_{i=1}^m \lambda_{ki} g^*_{ki}$ and where the equality follows from Eq. (2.3) and the fact that $\sum_{i=1}^m \lambda_{ki} = 1$. Therefore we get

$$\sum_{1 \leqslant k \leqslant n}^{1 \leqslant i \leqslant m} \mu(I_k \cap J_i) \|u^j_{ki} - g^*_{ki}\|^p \geqslant \sum_{k=1}^n \mu(I_k) \|u^j_k - \beta_k\|^p, \qquad j = 1, 2. \tag{2.5}$$

Hence, using Eqs. (2.1) then (1.1) and (2.5) then (2.4), we get

$$\begin{aligned} &|((\mu^{1/p}(I_k)(u^1_k - g_k))^n_{k=1}, (\mu^{1/p}(I_k)(u^2_k - g_k))^n_{k=1})|_{p,n} \\ &\qquad \leqslant |((\mu^{1/p}(I_k)(u^1_k - \beta_k))^n_{k=1}, (\mu^{1/p}(I_k)(u^2_k - \beta_k))^n_{k=1})|_{p,n} \\ &\qquad \leqslant |((\mu^{1/p}(I_k \cap J_i)(u^1_{ki} - g^*_{ki}))^{n,m}_{k,i=1}, (\mu^{1/p}(I_k \cap J_i)(u^2_{ki} - g^*_{ki}))^{n,m}_{k,i=1})|_{p,nm} \\ &\qquad \leqslant |((\mu^{1/p}(I_k \cap J_i)(u^1_{ki} - h_{ki}))^{n,m}_{k,i=1}, (\mu^{1/p}(I_k \cap J_i)(u^2_{ki} - h_{ki}))^{n,m}_{k,i=1})|_{p,nm}. \end{aligned}$$

In other words,

$$N(\|f_1-g\|_p, \|f_2-g\|_p) \leqslant N(\|f_1-h\|_p, \|f_2-h\|_p)$$

for all simple functions $h \in L^p(I, G)$ and, consequently, for all functions $h \in L^p(I, G)$, since the set of simple functions is dense in $L^p(I, G)$. The proof is complete. ▮

COROLLARY 1. *If G is reflexive, then every pair of simple functions $f_1, f_2 \in L^p(I, E)$ admits a best N-simultaneous approximation g from $L^p(I, G)$.*

From the proof of the theorem we obtain:

Remark 3. If, for every $n \geqslant 1$, G^n is N-simultaneously proximinal in E^n, then every pair of simple functions in $L^p(I, E)$ admits a simple function in $L^p(I, G)$ as an N-simultaneous approximation.

In the special case where N is the p-norm on R^2, we obtain a stronger result than that of Theorem 1:

THEOREM 2. *If $N(x_1, x_2) := (|x_1|^p + |x_2|^p)^{1/p}$ and if $L^p(I, D^1)$ is proximinal in $L^p(I, E^2)$, then $L^p(I, G)$ is N-simultaneously proximinal in $L^p(I, E)$.*

Proof. First we note that

$$\begin{aligned} N(\|f_1\|_p, \|f_2\|_p) &= \left(\int_I \|f_1(s)\|^p \, d\mu + \int_I \|f_2(s)\|^p \, d\mu\right)^{1/p} \\ &= \left(\int_I (\|f_1(s)\|^p + \|f_2(s)\|^p) \, d\mu\right)^{1/p} \\ &= \left(\int_I [N(\|f_1(s)\|, \|f_2(s)\|)]^p \, d\mu\right)^{1/p}. \end{aligned}$$

This implies that $[L^p(I, E)]^2$ is isometric to $L^p(I, E^2)$ and that $D^1_{L^p(I, G)} := \{(g, g) : g \in L^p(I, G)\}$ is isometric to $L^p(I, D^1)$. Hence we obtain, from the assumption, that $D^1_{L^p(I, G)}$ is proximinal in $[L^p(I, E)]^2$. The theorem now follows from the fact that $L^p(I, G)$ is N-simultaneously proximinal in $L^p(I, E)$ if and only if $D^1_{L^p(I, G)}$ is proximinal in $[L^p(I, E)]^2$. End of the proof. ▮

On the question of proximinality of $L^p(I, H)$ in $L^p(I, X)$, where X is a Banach space and H is a closed subspace satisfying some conditions (in our case $X = E^2$ and $H = D^1$), many results have been established by various authors, e.g., [2, 4–6, 8, 12] to mention a few. For some of the strongest results on this question, we refer the reader to [12] and [8]. Therefore, one can obtain several corollaries from Theorem 2. In particular, if G is reflexive then, by Remark 1, D^1 is reflexive and, consequently, $L^p(I, D^1)$ is proximinal in $L^p(I, E^2)$, [12].

Note that if G is reflexive and $1 < p < \infty$, then it follows, by Remark 1 and by [3, IV.1. Corollary 2], that $L^p(I, G)$ and $L^p(I, D^1)$ are reflexive. Therefore, for $p > 1$, we obtain directly, by Lemma 1, the following more general result than those of Corollary 1 and Theorem 2:

THEOREM 3. *If G is reflexive and $1 < p < \infty$, then $L^p(I, G)$ is N-simultaneously proximinal in $L^p(I, E)$.*

The case where $p = 1$ and N is arbitrary is more difficult and will be studied in Section 3.

3. BEST SIMULTANEOUS APPROXIMATION IN $L^1(I, E)$

First, we establish some preliminary results needed for the proof of our main theorem:

LEMMA 2. *If $|x_j| < |y_j|$ in R, $j = 1, 2$, then $N(x_1, x_2) < N(y_1, y_2)$.*

Proof. From the assumption we get that there exist $\alpha_1, \alpha_2 \in [0, 1)$ such that $|x_j| = \alpha_j |y_j|$, $j = 1, 2$. Let $\lambda := \max\{\alpha_1, \alpha_2\}$. Then $\lambda < 1$ and $|x_j| \leqslant \lambda\, |y_j|$, $j = 1, 2$.

Therefore by Eq. (1.1) we get

$$N(x_1, x_2) \leqslant N(\lambda y_1, \lambda y_2) = \lambda N(y_1, y_2).$$

But $\lambda < 1$ and from the assumption $N(y_1, y_2) > 0$. Therefore $N(x_1, x_2) < N(y_1, y_2)$. ∎

LEMMA 3. *If $g \in L^p(I, G)$ is a best N-simultaneous approximation from $L^p(I, G)$ of the pair of elements $f_1, f_2 \in L^p(I, E)$ then, for every measurable subset A of I and every $h \in L^p(I, G)$,*

$$\int_A \|f_{j_o}(s) - g(s)\|^p \, d\mu \leqslant \int_A \|f_{j_o}(s) - h(s)\|^p \, d\mu,$$

for some $j_o \in \{1, 2\}$.

Proof. If $\mu(A)=0$ then there is nothing to prove. Suppose that, for some A satisfying $\mu(A)>0$ and for some $h_o \in L^p(I, G)$, the inequality does not hold for $j=1$ and for $j=2$. Now, define $g_o \in L^p(I, G)$ by

$$g_o(s) := \begin{cases} g(s) & \text{if } s \in I-A \\ h_o(s) & \text{if } s \in A. \end{cases}$$

Then we have, for $j=1, 2$,

$$\begin{aligned}
&\left[\int_I \|f_j(s)-g_o(s)\|^p \, d\mu\right]^{1/p} \\
&\quad = \left[\int_A \|f_j(s)-h_o(s)\|^p \, d\mu + \int_{I-A} \|f_j(s)-g(s)\|^p \, d\mu\right]^{1/p} \\
&\quad < \left[\int_A \|f_j(s)-g(s)\|^p \, d\mu + \int_{I-A} \|f_j(s)-g(s)\|^p \, d\mu\right]^{1/p} \\
&\quad = \left[\int_I \|f_j(s)-g(s)\|^p \, d\mu\right]^{1/p}.
\end{aligned}$$

This together with Lemma 2 imply that

$$N(\|f_1-g_o\|_p, \|f_2-g_o\|_p) < N(\|f_1-g\|_p, \|f_2-g\|_p)$$

which contradicts the fact that g is a best N-simultaneous approximation from $L^p(I, G)$ of the pair of elements f_1, f_2. ∎

As a corollary we get:

COROLLARY 2. *If g is a best N-simultaneous approximation from $L^p(I, G)$ of the pair of elements $f_1, f_2 \in L^p(I, E)$ then, for every measurable subset A of I,*

$$\int_A \|g(s)\|^p \, d\mu \leqslant 2 \max\left\{\int_A \|f_1(s)\|^p \, d\mu, \int_A \|f_2(s)\|^p \, d\mu\right\}.$$

Proof. Since, for $j=1, 2$,

$$\left[\int_A \|g(s)\|^p \, d\mu\right]^{1/p} \leqslant \left[\int_A \|f_j(s)-g(s)\|^p \, d\mu\right]^{1/p} + \left[\int_A \|f_j(s)\|^p \, d\mu\right]^{1/p},$$

we obtain, by using Lemma 3 with $h := 0$, that for some $j_o \in \{1, 2\}$

$$\left[\int_A \|g(s)\|^p \, d\mu\right]^{1/p} \leqslant 2\left[\int_A \|f_{j_o}(s)\|^p \, d\mu\right]^{1/p}$$
$$\leqslant 2 \max\left\{\left[\int_A \|f_1(s)\|^p \, d\mu\right]^{1/p}, \left[\int_A \|f_2(s)\|^p \, d\mu\right]^{1/p}\right\},$$

which completes the proof. ∎

We note that, as a corollary of Lemma 3, we get that, if $g \in G^n$ is a best N-simultaneous approximation from G^n of the pair of elements u^1, $u^2 \in E^n$ then, for each $h \in G^n$ and each k, $1 \leqslant k \leqslant n$,

$$\text{either} \quad \|u_k^1 - g_k\| \leqslant \|u_k^1 - h_k\| \qquad \text{or} \qquad \|u_k^2 - g_k\| \leqslant \|u_k^2 - h_k\|.$$

Hence, for every $J \subset \{1, 2, ..., n\}$,

$$\sum_{k \in J} \|g_k\|^p \leqslant 2 \max\left\{\sum_{k \in J} \|u_k^1\|^p, \sum_{k \in J} \|u_k^2\|^p\right\}.$$

We are now ready to establish the analogue of Theorem 3 for $L^1(I, E)$:

THEOREM 4. *If G is reflexive then $L^1(I, G)$ is N-simultaneously proximinal in $L^1(I, E)$.*

Proof. Let f_1, $f_2 \in L^1(I, E)$ and let $\{f_{jn}\}_{n=1}^{\infty}$, $j = 1, 2$, be two sequences of simple functions in $L^1(I, E)$ satisfying

$$\lim_{n \to \infty} \|f_j - f_{jn}\|_1 = 0, \qquad j = 1, 2.$$

By Corollary 1 we obtain, for each $n \geqslant 1$, that the pair of simple functions f_{1n}, f_{2n} admits a best N-simultaneous approximation g_n from $L^p(I, G)$. Hence we have, for each $n \geqslant 1$,

$$N(\|f_{1n} - g_n\|_1, \|f_{2n} - g_n\|_1) \leqslant N(\|f_{1n} - h\|_1, \|f_{2n} - h\|_1), \tag{3.1}$$

for every $h \in L^1(I, G)$. By Corollary 3, we obtain that

$$\int_A \|g_n(s)\| \, d\mu \leqslant 2 \max\left\{\int_A \|f_{1n}(s)\| \, d\mu, \int_A \|f_{2n}(s)\| \, d\mu\right\}$$

for every $n \geqslant 1$. It follows, since both $\{f_{1n}\}_{n=1}^{\infty}$ and $\{f_{2n}\}_{n=1}^{\infty}$ are uniformly integrable, i.e.,

$$\sup\left\{\sup_n \left\{\int_A \|f_{jn}(s)\|\, d\mu\right\} : A \subset I,\ \mu(A) \leqslant \varepsilon\right\} \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

and since $f_{jn} \to f_j$ in $L^1(I, E)$, $j = 1, 2$, that the sequence $\{g_n\}_{n=1}^{\infty}$ in $L^1(I, G)$ is bounded and uniformly integrable. Hence, since G is reflexive, we obtain, by Dunford's theorem [3], that $\{g_n\}_{n=1}^{\infty}$ is relatively weakly compact in $L^1(I, E)$. Therefore, there exists a subsequence, say $\{g_n\}_{n=1}^{\infty}$, which converges weakly to some element $g \in L^1(I, E)$. It follows, since $L^1(I, G)$ is closed and convex hence weakly closed, that $g \in L^1(I, G)$. It follows from Eq. (1.1) that $N(\|\cdot\|_1, \|\cdot\|_1)$ is convex and continuous, and hence weakly lower semicontinuous, on $L^1(I, E)$. This together with Eq. (3.1) imply that, for every $h \in L^1(I, G)$,

$$\begin{aligned} N(\|f_1 - g\|_1, \|f_2 - g\|_1) &\leqslant \liminf_n N(\|f_{1n} - g_n\|_1, \|f_{2n} - g_n\|_1) \\ &\leqslant \liminf_n N(\|f_{1n} - h\|_1, \|f_{2n} - h\|_1) \\ &= N(\|f_1 - h\|_1, \|f_2 - h\|_1). \end{aligned}$$

Therefore g is a best N-simultaneous approximation from $L^p(I, G)$ of the pair $f_1, f_2 \in L^1(I, E)$. ∎

Finally, we note the following:

Remark 4. It follows immediately that all the results and proofs in this paper are valid in the case where N is a norm on R^M, $M \geqslant 2$, satisfying

$$N(x) \leqslant N(y), \qquad \mathit{if} \quad |x_i| \leqslant |y_i|, \qquad 1 \leqslant i \leqslant M.$$

In this case, $g \in G$ is said to be a best N-simultaneous approximation from G of the elements $u^1, u^2, \ldots, u^M \in E$ if, for every $h \in G$,

$$N(\|u^1 - g\|, \|u^2 - g\|, \ldots, \|u^M - g\|) \leqslant N(\|u^1 - h\|, \|u^2 - h\|, \ldots, \|u^M - h\|).$$

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